Three Papers on Takums

Laslo Hunhold[1]

[1]Parallel and Distributed Systems Group, University of Cologne, Germany

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Fundamental innovations and improvements are not only possible, but the norm.

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- Not limited to 8, 16, 32, 64, etc. bits of precision

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  - this creates a zoo of formats, which is complicated for hardware implementers and users

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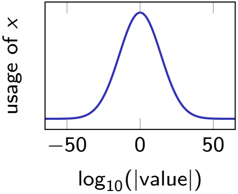

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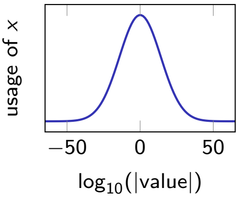

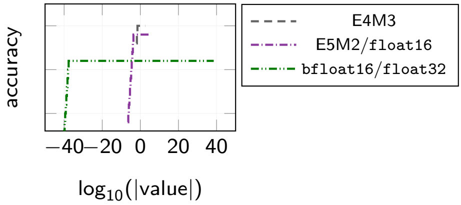

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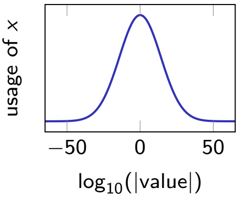

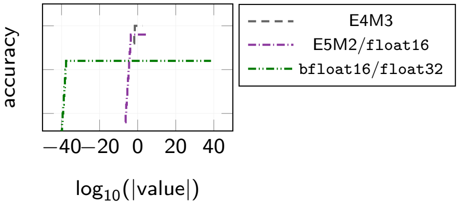

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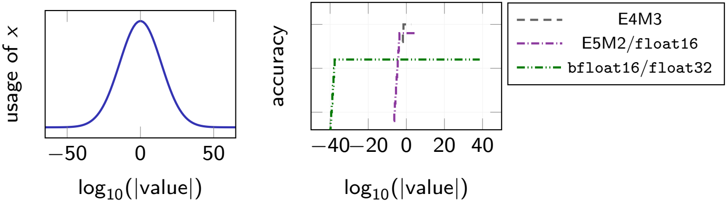

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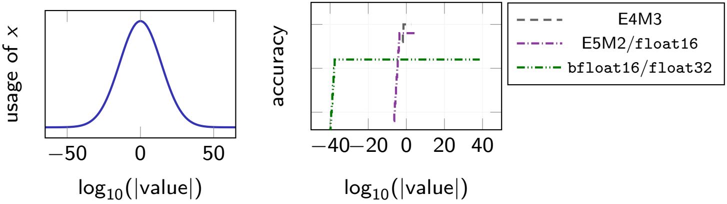

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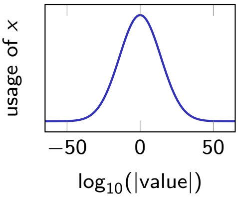

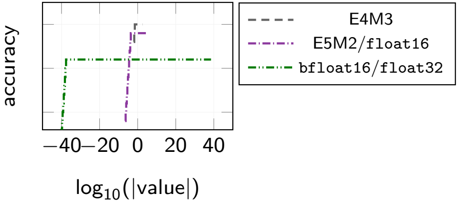

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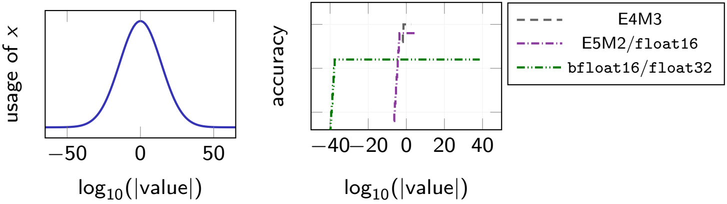

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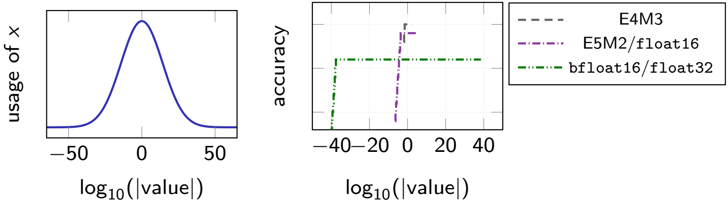

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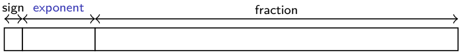

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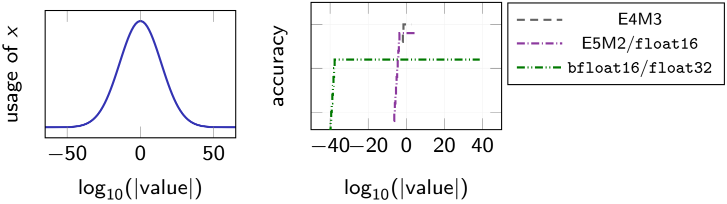

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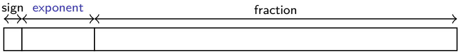

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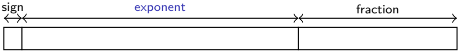

Posits

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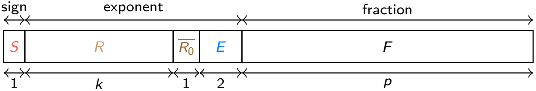

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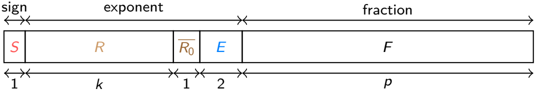

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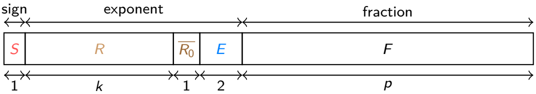

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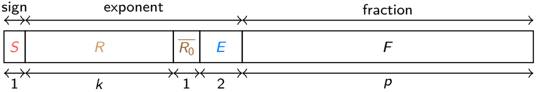

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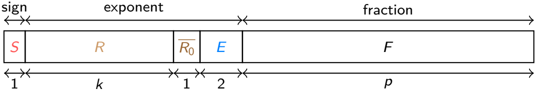

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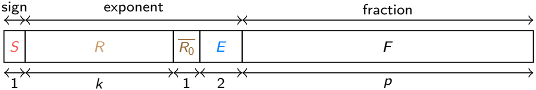

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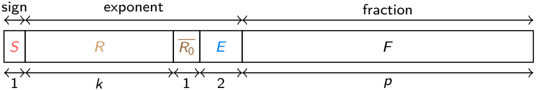

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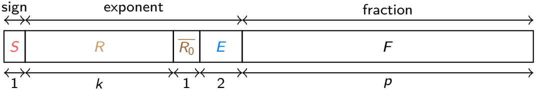

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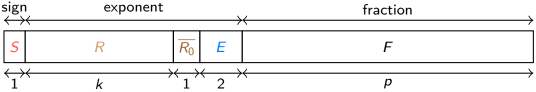

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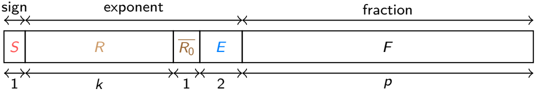

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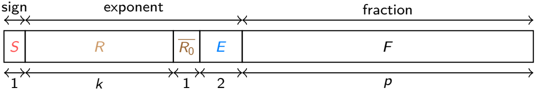

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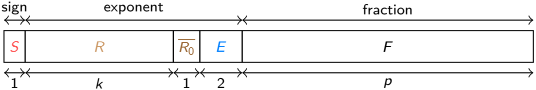

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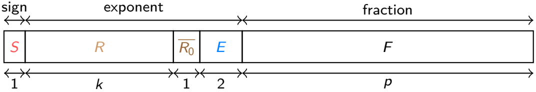

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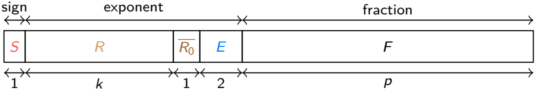

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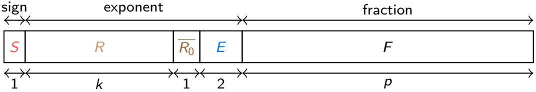

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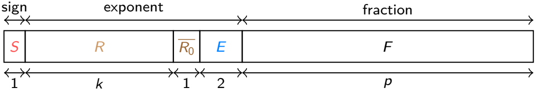

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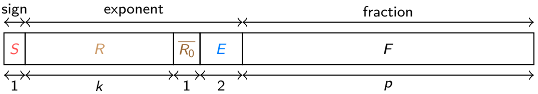

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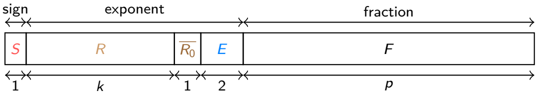

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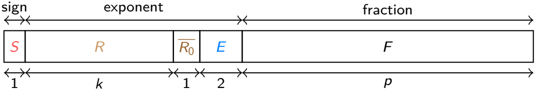

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  - Symmetry with two's complement integers (unsigned zero of all zero bits, negation, ordering, sign)
  - Defined for all n, conversion between lengths simple rounding or expansion

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Posit Examples

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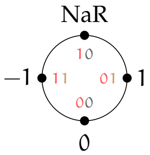
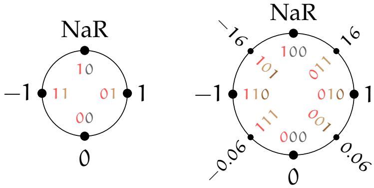


(a) $n = 2$

Posit Examples



(a) $n = 2$ (b) $n = 3$

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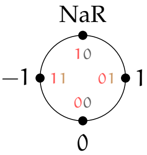

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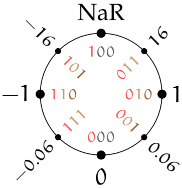

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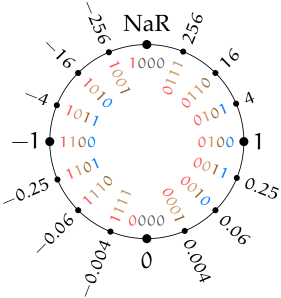
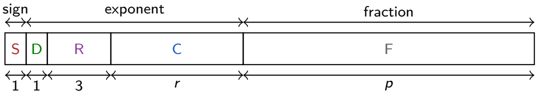


(c) $n = 4$

Takums



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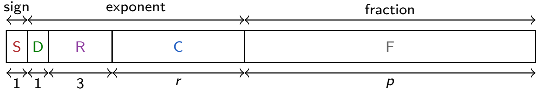
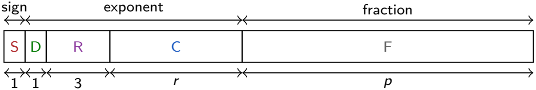


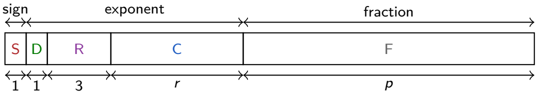
- Goals

Takums



- Goals
  - Preserve useful posit properties (two's complement symmetry, etc.)

Takums



- Goals
  - Preserve useful posit properties (two’s complement symmetry, etc.)
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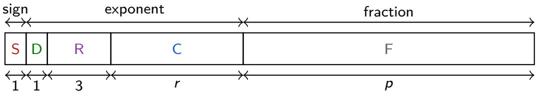

- Goals
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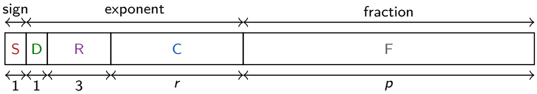
Takums



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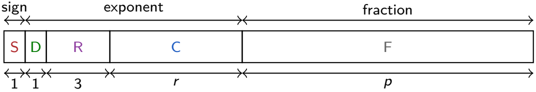

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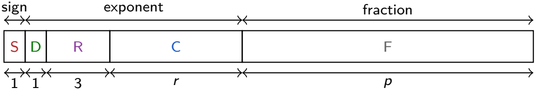
  - Exponent value has implicit leading 1, subtract 1 for range 0-254

value	0	1	2	3	4	...	254
value bits	1	10	11	100	101	...	11111111
encoding	000	0010	0011	01000	01001	...	1111111111

Takums



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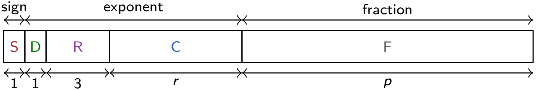

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- Improvements over posits

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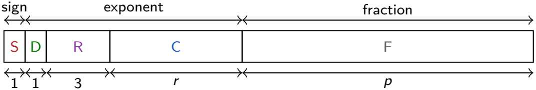

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  - 3-bit regime (0-7) encodes length, followed by 0 to 7 exponent bits.
  - Exponent value has implicit leading 1, subtract 1 for range 0-254

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encoding	000	0010	0011	01000	01001	...	1111111111

  - Separate 'direction' bit D for exponent value sign
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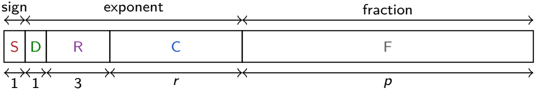
  - Exponent is at most 11 bits long ($\rightarrow$ simpler hardware)

Takums



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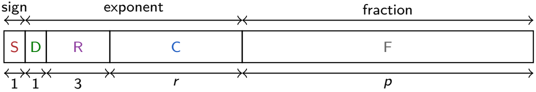

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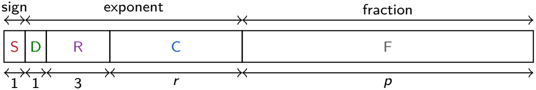

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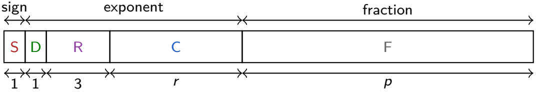

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$\downarrow$

11111111111 (11b)

Takum Examples

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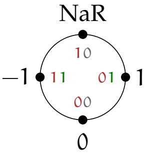
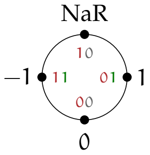


(a) $n = 2$

Takum Examples



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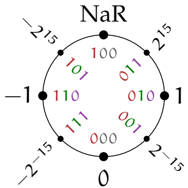

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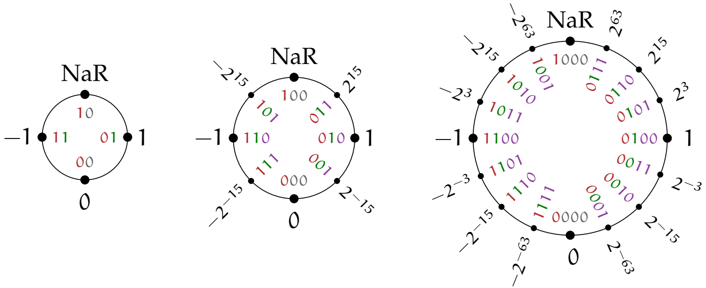
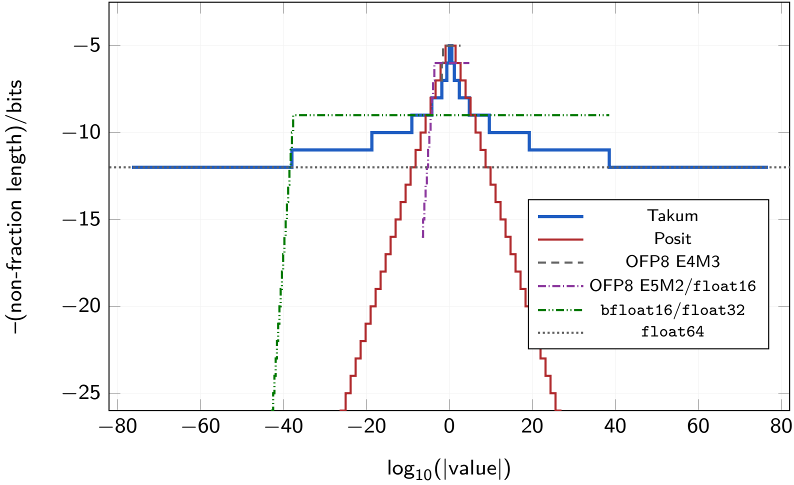


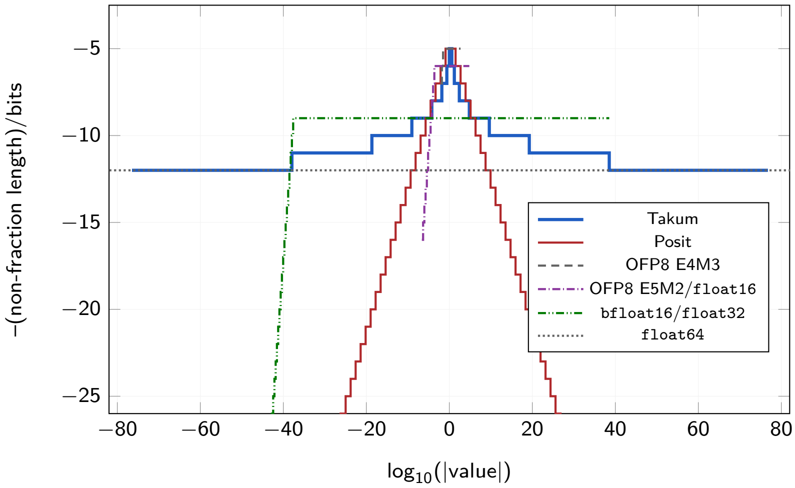
(a) $n = 2$ (b) $n = 3$ (c) $n = 4$

Tapering



−5
−10
−15
−20
−25
−(non-fraction length)/bits
−80
−60
−40
−20
0
20
40
60
80
log10(|value|)
Takum
Posit
OFP8 E4M3
OFP8 E5M2/float16
bfloat16/float32
float64

Tapering



Posit: Linear tapering, drop-off at the edges

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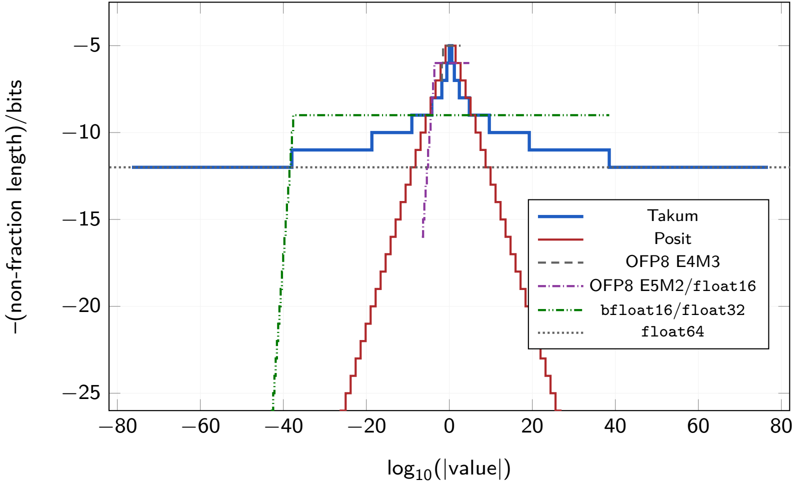
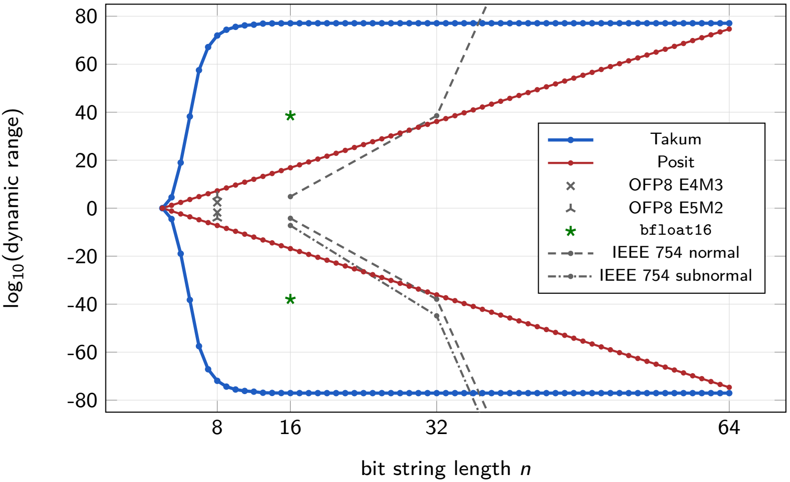


Posit: Linear tapering, drop-off at the edges
Takum: Logarithmic tapering, no drop-off at the edges

Dynamic Range



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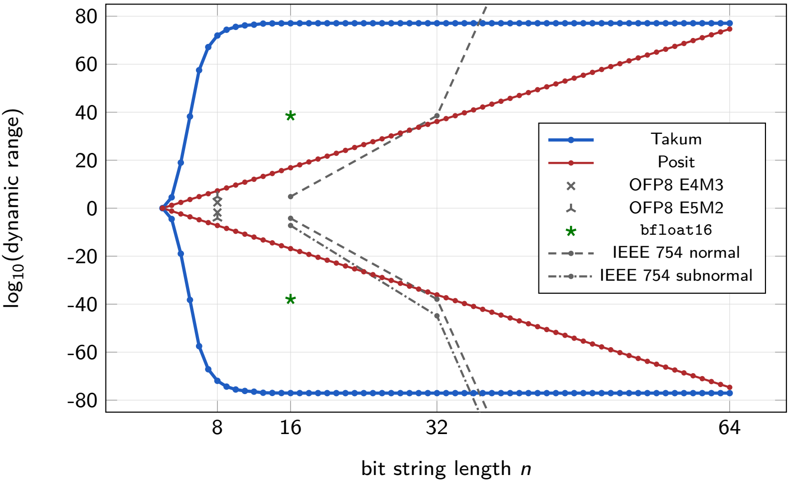

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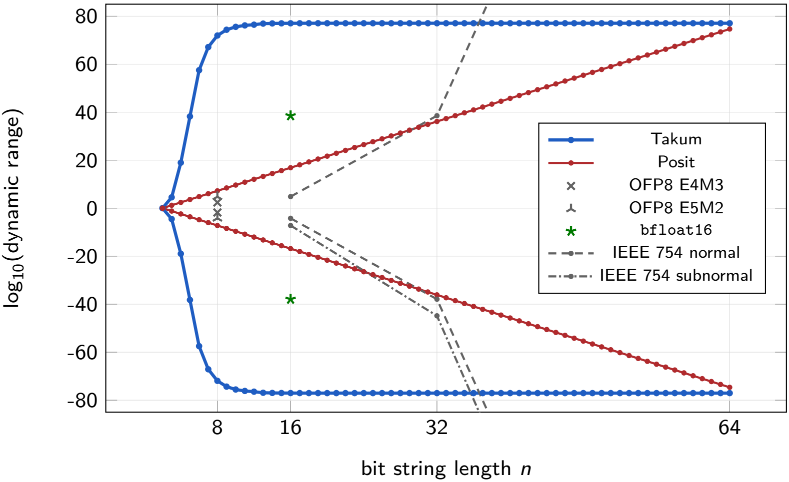

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First Paper

Integer Representations in IEEE 754, OFP8, Bfloat16, Posit, and Takum Arithmetics

Integer Representations

Motivation and Approach

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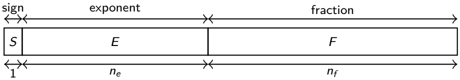
- Different question: Given an integer, how many bits do I need to represent it as a posit or takum?

Integer Representations

IEEE 754 Floating-Point Numbers



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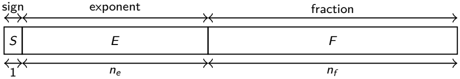

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Integer Representations

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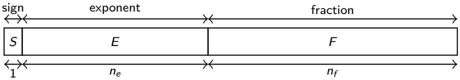

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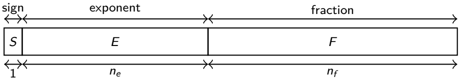

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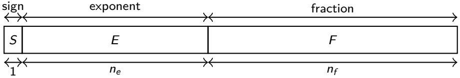

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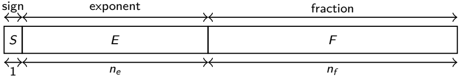

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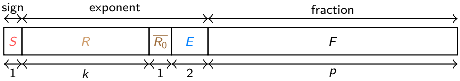
Integer Representations

Posits



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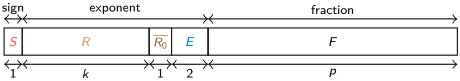

- Let $m \in \mathbb{Z} \setminus \{0\}$ with $v := 1 + \lfloor \log_2(|m|) \rfloor$ bits and $w := \max_{i \in \mathbb{N}_0} \left(2^i \mid m\right)$ trailing zeros in $|m|$'s binary representation. There exists an $M \in \{0,1\}^\ell$ with $\pi(M) = m$ and

$$\ell := \left\lfloor \frac{5(v+3)}{4} - w \right\rfloor - (w = v - 1) \cdot \begin{cases} 3 & v \in 4\mathbb{N}_0 + 1 \\ 1 & v \in 4\mathbb{N}_0 + 3, \end{cases}$$

which is the shortest possible representation.

Integer Representations

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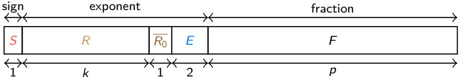

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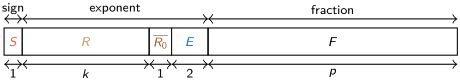

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Integer Representations
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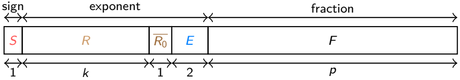

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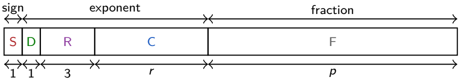
Integer Representations

Takums



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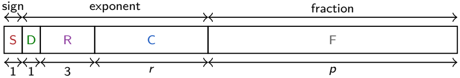

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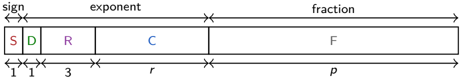

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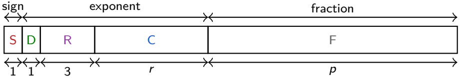

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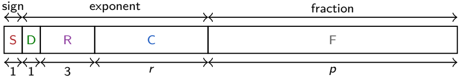

- Let $m \in \mathbb{Z} \setminus \{0\}$ with $|m| \leq 2^{254}$, $v := 1 + \lfloor \log_2(|m|) \rfloor$ bits and $w := \max_{i \in \mathbb{N}_0} \left(2^i \mid m\right)$ trailing zeros in $|m|$'s binary representation. There exists an $M \in \{0,1\}^\ell$ with $\tau(M) = m$ and

$$\begin{aligned}\ell := \lfloor 4 + v + \log_2(v) - w \rfloor - \\ & (w = v - 1) \cdot \max_{i \in \{0,\dots,\lfloor \log_2(v) \rfloor\}} \left(2^i \mid v - 2^{\lfloor \log_2(v) \rfloor}\right) - \\ & (v \in 2^{\mathbb{N}_0}) \cdot \max_{i \in \{0,\dots,3\}} \left(2^i \mid \lfloor \log_2(v) \rfloor\right),\end{aligned}$$

  which is the shortest possible representation.
- Proof: Just like for posits, only harder. $\square$
- Let $n \in \mathbb{N}_5$. It holds for all $m \in \mathbb{Z}$ with $|m| \leq 2^{254}$ and

$$|m| \leq 2^{\left\lceil \frac{W_0\left(2^{n-3}\ln(2)\right)}{\ln(2)} - 1 \right\rceil},$$

  where W_0 is the principal branch of the Lambert W function, that there exists an $M \in \{0,1\}^n$ with $m = \tau(M)$.
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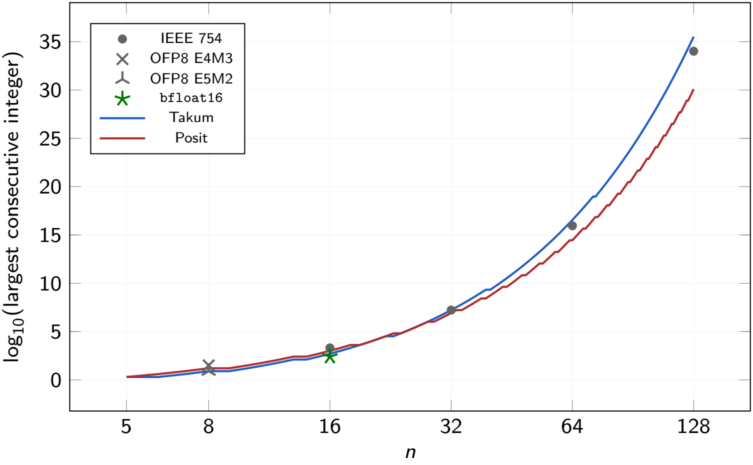

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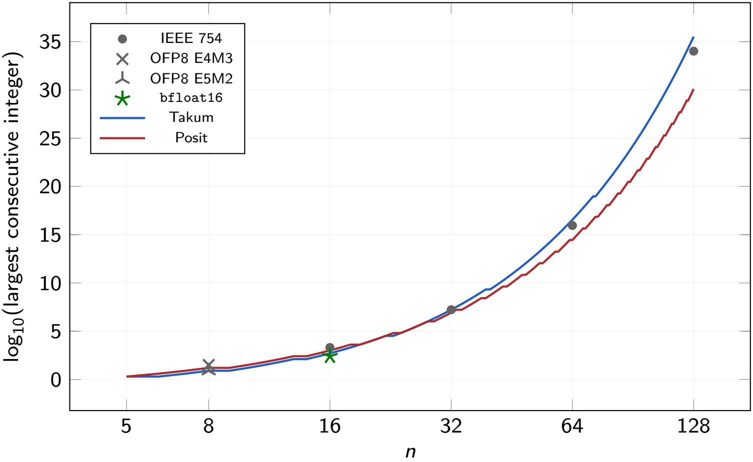

Posits slightly better than takums for $n < 20$

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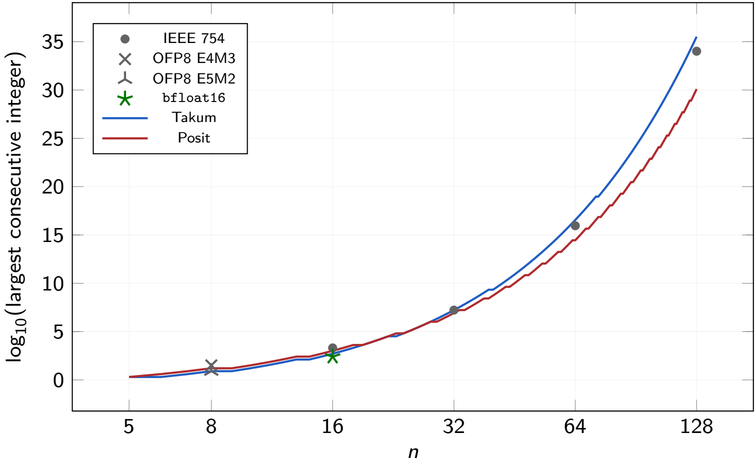

Posits slightly better than takums for $n < 20$

Posits scale worse, takums scale better than IEEE 754 (both outperform `bfloat16`)

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Conclusion and Outlook

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- Outlook: B-Posits

Second Paper

Design and Implementation of a Takum Arithmetic Hardware Codec

Takum FPGA Codec

Overview and Decoder Design

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- Current state of posit APUs: Decode to IR, compute with IR, encode from IR

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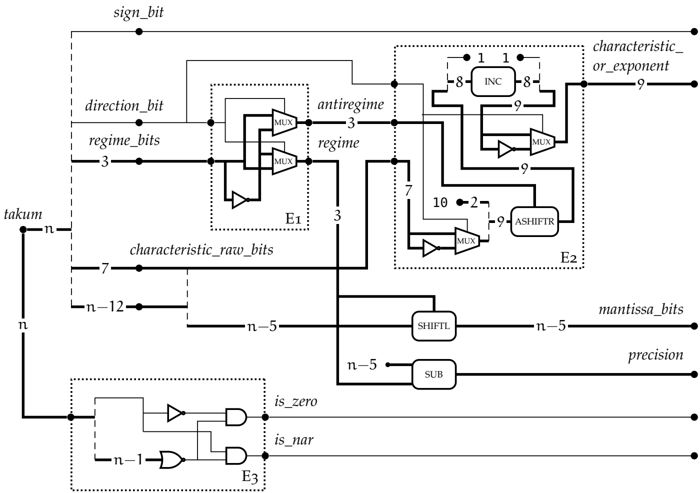
- Posits: Exponent can span entire width, requires leading-zero-counter, shifter scales in n

Takum FPGA Codec

Decoder Schematic



regime/antiregime determinator (E1), characteristic/exponent determinator (E2), special case detector (E3)

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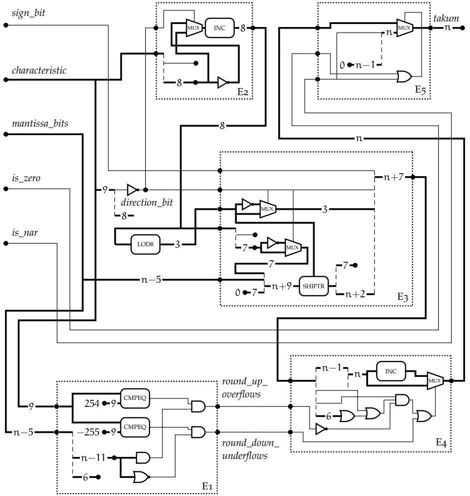
- Generate $(n+7)$-bit *extended takum* as a rounding candidate to avoid double-rounding

Takum FPGA Codec

Encoder Schematic



underflow/overflow predictor (E1), characteristic precursor determinator (E2), extended takum generator (E3), rounder (E4), output driver (E5)

Takum FPGA Codec

Benchmarks

[4] Raul Murillo et al. ‘Comparing different decodings for posit arithmetic’. CoNGA, 2022

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Comparison against the state of the art (FloPoCo, 2018-2023)[4], target 200MHz

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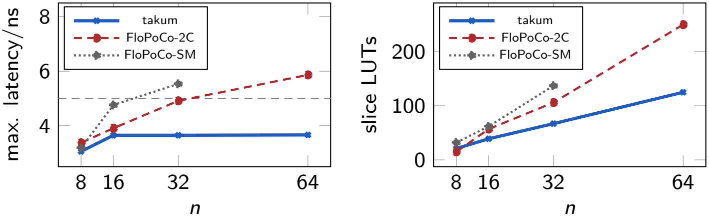

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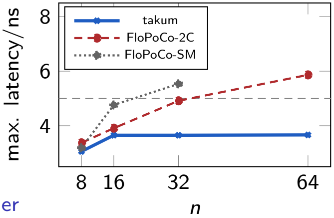

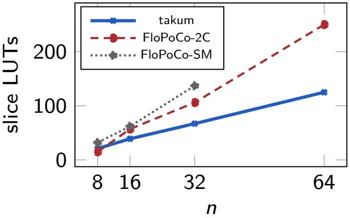

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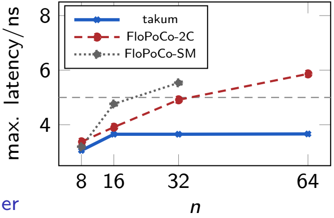

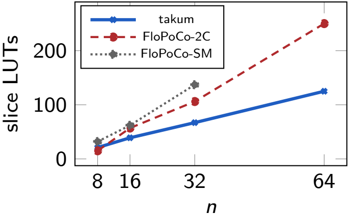

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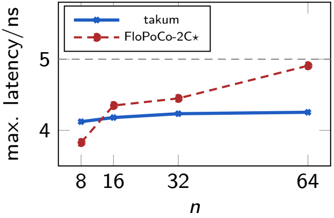

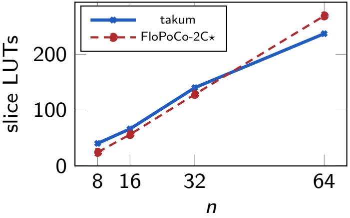

⋆: requires external rounding information

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SC25 Research Track (with James Quinlan, Stefan Wesner)

Numerical Performance of the Implicitly Restarted Arnoldi Method in OFP8, Bfloat16, Posit, and Takum Arithmetics

Tuesday, 3:52-4:15 p.m., Hall 274

Third Paper

Tekum: Balanced Ternary Tapered Precision Real Arithmetic

Tekums

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- No prior works, except `ternary27` by O'Hare (unpublished Hackaday project, inspired by IEEE 754, tritfield approach...), archived on Zenodo for posterity

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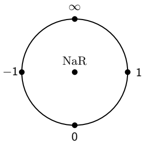

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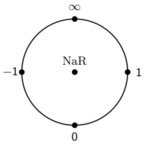

- Powers of three are odd, and $4r + 5$ (for $r \in \mathbb{N}_0$) is also odd (reason why posits/takums have only one of ∞ or NaR)

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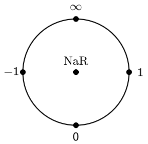

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Tekums

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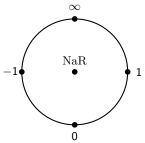

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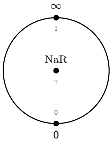

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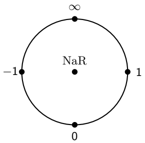

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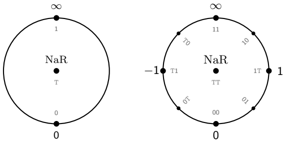

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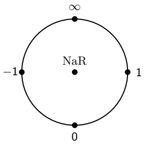

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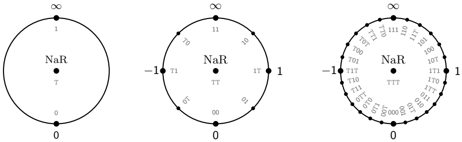

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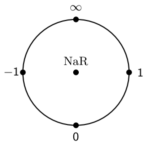

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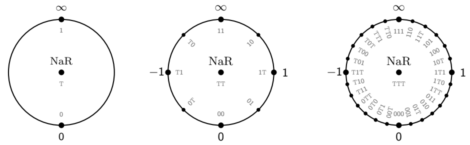

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- Solution: It holds $4 \mid (3^{2n} - 5)$ and $4 \nmid (3^n - 4)$! Dropping one special case will never work, but even ns always work.

Tekums

Second Filter - Misfit Tool

Tekums

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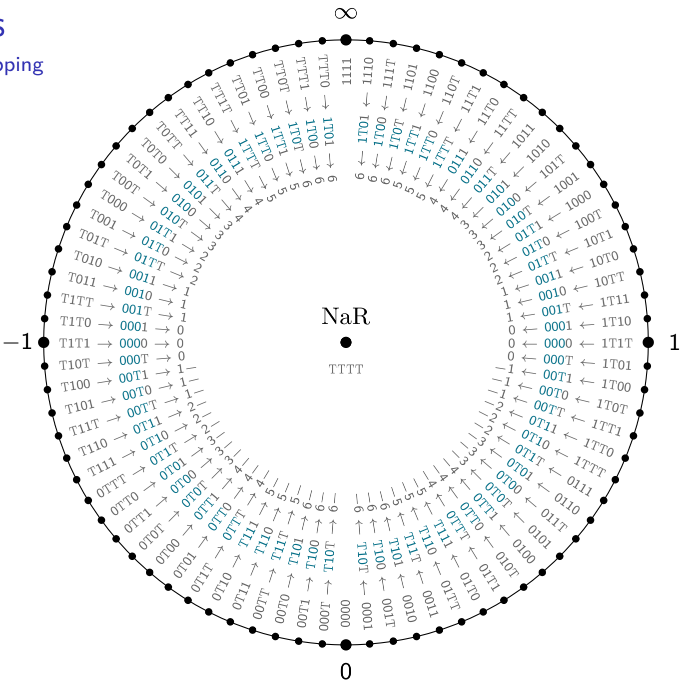
- Works the same for negative numbers, define *anchor function* $\mathrm{anc}(\boldsymbol{t}) := |t| - 1\mathrm{T} \cdots 1\mathrm{T}$.
- First three trits of anchor are in the range $\mathrm{T}1\mathrm{T} = -7$ to $1\mathrm{T}1 = 7$. Use the takum approach, designate first three trits to be regime (two would be too few, four would be too many)

Tekums

Regime Mapping



Tekums

Third Filter - Excess

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- Regime values r are in the range -7 to 7
- Map regime modulus to number of exponent trits $\boldsymbol{e}$ following regime trits, apply bias for proper ranges

r	0	1	2	3	4	5	6	7
$\lvert r \rvert$	0	1	2	3	4	5	6	7
$\mathrm{int}(\boldsymbol{e})$	$0\,..\,0$	$-1\,..\,1$	$-4\,..\,4$	$-13\,..\,13$	$-40\,..\,40$	$-121\,..\,121$	$-364\,..\,364$	$-1093\,..\,-547$
b	0	2	8	26	80	242	728	2186
e	$0\,..\,0$	$1\,..\,3$	$4\,..\,12$	$13\,..\,39$	$40\,..\,120$	$121\,..\,363$	$364\,..\,1092$	$1093\,..\,1639$

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- Try other mappings (see paper); best choice:

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$\operatorname{int}(\boldsymbol{e})$	$0..0$	$0..0$	$0..0$	$-1..1$	$-4..4$	$-13..13$	$-40..40$	$-121..-61$
b	0	1	2	4	10	28	82	244
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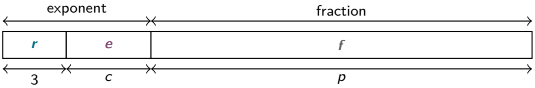
- $3^{183} \approx 10^{87}$, which is a good dynamic range

Tekums

Definition



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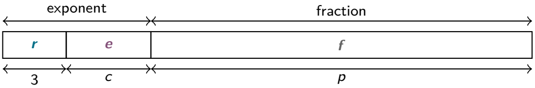

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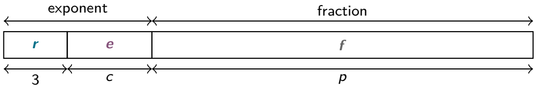

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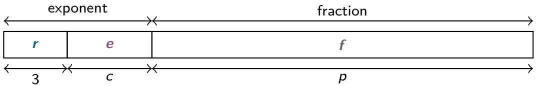

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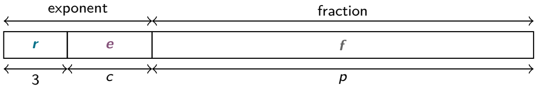

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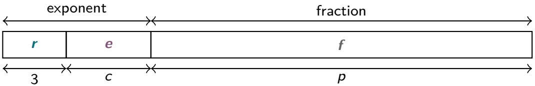

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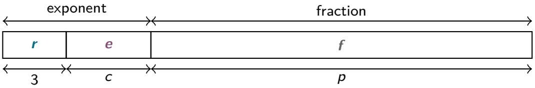

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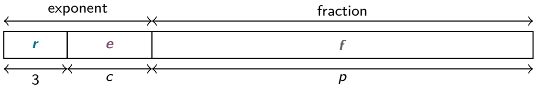

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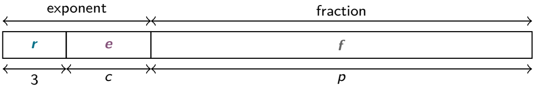

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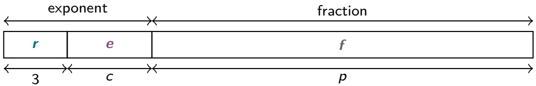

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Tekums

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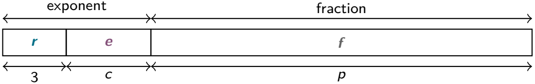

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- Obtain the tekum value

$$
\theta_n(\boldsymbol{t}) := \begin{cases} \mathrm{NaR} & \boldsymbol{r} \mathbin{+\!\!+} \boldsymbol{e} \mathbin{+\!\!+} \boldsymbol{f} = \mathrm{T} \cdots \mathrm{T} \\ 0 & \boldsymbol{r} \mathbin{+\!\!+} \boldsymbol{e} \mathbin{+\!\!+} \boldsymbol{f} = 0 \cdots 0 \\ \infty & \boldsymbol{r} \mathbin{+\!\!+} \boldsymbol{e} \mathbin{+\!\!+} \boldsymbol{f} = 1 \cdots 1 \\ s \cdot (1 + f) \cdot 3^e & \text{otherwise} \end{cases}
$$

Tekums

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- Only 3 trit fields instead of 5 bit fields (posits, takums)

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Tekums

Properties

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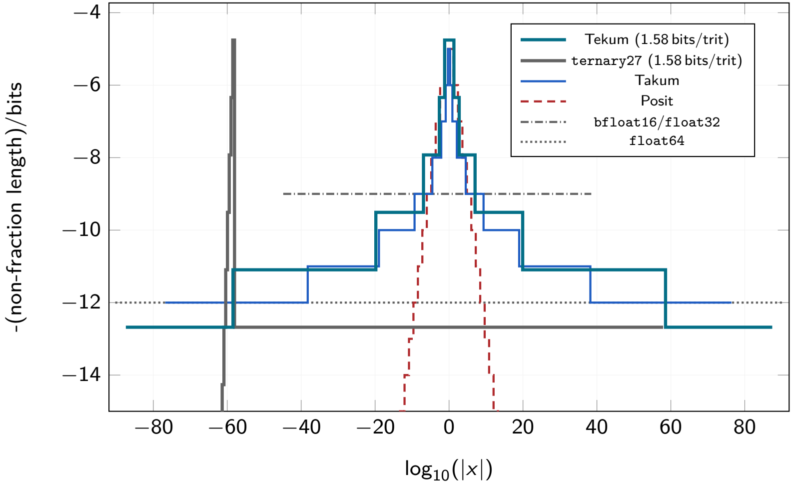
How do we compare tekums against binary number formats?

Tekums

Accuracy



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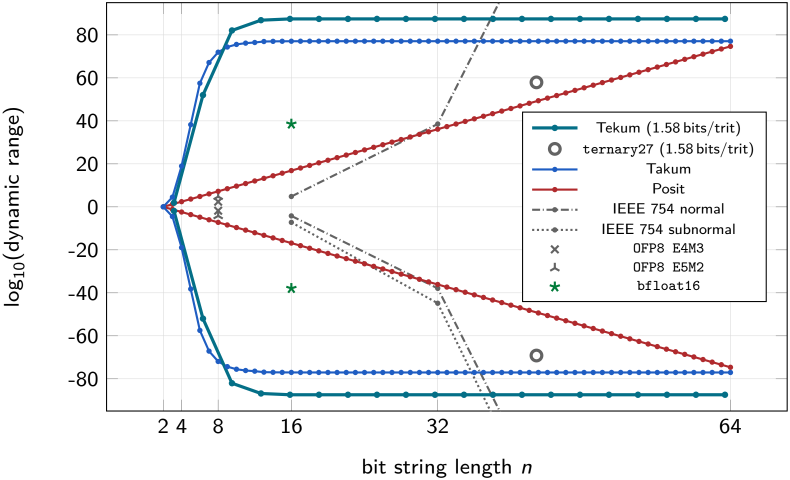
Dynamic Range



Tekums

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What will the arithmetic of the future be?
Tailored for AI or general-purpose?

takum
arithmetic